On the ‘transitivity’ of consequence relations

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Abstract

A binary relation R on a set S is *transitive* iff for all $a,b,c \in S$, if aRb and bRc, then aRc. This almost never applies to the relations logicians tend to think of as *consequence relations*; where such relations are relations on a set at all, they are rarely transitive. Yet it is common to hear consequence relations described as ‘transitive’, and to see rules imposed to ensure ‘transitivity’ of these relations. This article attempts to clarify the situation.

Keywords: Transitivity, consequence relations, abstract logic.

1 Introduction

After briefly substantiating the claims in the abstract, this article focuses on exploring a number of different properties of consequence relations that have traveled under the name ‘transitivity’, mapping the implications among them. From here forward, I will use ‘transitive’ and ‘transitivity’ very little, and only in their standard relation-theoretic sense: a relation R is *transitive* iff it is a binary relation on a set S such that for any $a,b,c \in S$, if aRb and bRc, then aRc.

Many familiar consequence relations, however, are not relations on a set at all, but instead relate *sets* of formulas (collections of premises) to *single* formulas (conclusions). That is, where $\mathcal{F}$ is the set of formulas under consideration, such a relation is a relation between $\wp(\mathcal{F})$ and $\mathcal{F}$. Following [9, §1.21], I will say these relations work in the ‘SET-FMLA framework’. Such a relation is not the right kind of thing to be transitive. Of course, these relations can, and frequently do, exhibit a number of properties that are reminiscent of transitivity in various ways. This article will explore some of the variety among these properties.

I won’t restrict my attention to the SET-FMLA framework, though. I will in addition consider consequence relations in the SET-SET framework. In this framework, consequence relations really are binary relations on a single set: the set $\wp(\mathcal{F})$. That is, they relate sets of formulas to sets of formulas. So they are at least the right *kind* of relation to be transitive.

Much research into SET-SET consequence relations (see e.g. [7, 9, 11, 17, 22, 23]) interprets the members of the set of conclusions as (in some sense) *different possibilities*. On this interpretation, arguments with *fewer* conclusions are stronger than those with more, since they *narrow down* more finely on a result. This is the interpretation I’ll focus on in what follows.[1]

These relations, too, are almost never transitive. Consider, for example, the SET-SET consequence relation $\vdash$ determined by classical logic, explored and defended in [11], among other places. This relation relates $\{A \vee B\}$ to $\{A, B\}$, and relates $\{A, B\}$ to $\{A \wedge B\}$, but does not relate $\{A \vee B\}$ to $\{A \wedge B\}$; it is thus not transitive. The reason is nothing particularly to do with classical logic; it is instead to do

[1] There is another way to approach SET-SET consequence relations; the members of the set of conclusions can be seen as (in some sense) *all* following. (See for example [3].) On this interpretation, arguments with *more* conclusions are stronger, since they show that more follows from the premises. This is one area of logic where ‘transitivity’ as applied to consequence relations typically means *transitivity*. The situation for these relations, then, is clear; they are transitive, when they are, in the ordinary sense. As such, they are not my focus here, and I will not consider this kind of SET-SET consequence relation further.

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with how *sets* of formulas are interpreted. As premises, they are meant *conjunctively*: as all available to be drawn on together in establishing conclusions. As conclusions, they are meant *disjunctively*: as jointly exhausting the space where the truth must lie, given the premises.[2] This difference in interpretation prevents linking valid SET-SET arguments together in the simple way guaranteed by transitivity; an attempt to apply transitivity to these SET-SET arguments would result in something like an equivocation.

This article only considers SET-FMLA and SET-SET consequence relations; I won't consider other options here. (See, again, [9, §1.21].)

2 Linking properties

In this section, I lay out the assumptions that will frame the article, and then present a catalogue of five properties that a SET-FMLA consequence relation might exhibit, and twelve properties that a SET-SET consequence relation might exhibit, all of which, I think, are recognizable as related to what logicians often mean by 'transitivity' as applied to these relations. These properties form the basis of the article, which fully maps the implications among arbitrary conjunctions of these properties.

Some notational preliminaries: I use capital Roman letters for formulas, and capital Greek letters (that are not also capital Romans) for sets of formulas. $\mathcal{F}$ is the set of formulas in the language under consideration, considered to be fixed throughout. When I talk of 'partitions' of a set, this should be understood to *include* partitions with an empty entry; e.g., $\langle \emptyset, \Sigma \rangle$ *is* a partition of Σ, on this usage.

I abbreviate freely in usual sequent-calculus ways, so, for example, '$\Gamma, A, \Sigma \vdash$' abbreviates '$\Gamma \cup \{A\} \cup \Sigma \vdash \emptyset$'. Note that this means '$\Gamma \vdash A$' should be interpreted differently depending on whether $\vdash$ is a SET-FMLA or SET-SET consequence relation. If the former, then 'A' here is no abbreviation; it names the formula A. If the latter, then 'A' here is an abbreviation, standing for the singleton set $\{A\}$.

2.1 Assumptions

I assume in places that the language $\mathcal{F}$ contains infinitely many formulas; its cardinality does not otherwise matter. I make no assumptions about the nature or structure of formulas; $\mathcal{F}$ can be any infinite set.

Consequence relations are often defined as relations that are '*reflexive*, *monotonic*, and *transitive*'. The final condition, of course, is the subject of this article, so I am certainly not assuming it.

I will, however, assume throughout the article that all consequence relations are *monotonic*. For SET-FMLA relations $\vdash$, this means that whenever $\Gamma \vdash A$, then $\Gamma, \Gamma' \vdash A$; for SET-SET relations $\vdash$, this means that whenever $\Gamma \vdash \Delta$, then $\Gamma, \Gamma' \vdash \Delta, \Delta'$.[3] This assumption matters a great deal; the situation is very different if this assumption is not imposed, and many of the results to follow would not hold without it.

A SET-FMLA relation $\vdash$ is *compact* iff whenever $\Gamma \vdash A$, then there is a finite $\Gamma_{\text{fin}} \subseteq \Gamma$ such that $\Gamma_{\text{fin}} \vdash A$; a SET-SET relation $\vdash$ is compact iff whenever $\Gamma \vdash \Delta$, then there are finite $\Gamma_{\text{fin}} \subseteq \Gamma$ and $\Delta_{\text{fin}} \subseteq \Delta$ such that $\Gamma_{\text{fin}} \vdash \Delta_{\text{fin}}$. In what follows, I will not require compactness in general, but I will keep track of compactness, and show what the effects of requiring compactness are. (This is the

[2] This is not to say, of course, that combining premises depends on some prior notion of conjunction, or that combining conclusions depends on some prior notion of disjunction.

[3] Unlike 'transitive' (and, it will emerge presently, 'reflexive'), 'monotonic' here *does* have its usual relation-theoretic sense, wrt the order $\subseteq$ on sets of formulas. (And, for SET-FMLA relations, the discrete order on formulas; that is, the order that relates each formula only to itself.)

reason for requiring $\mathcal{F}$ to be infinite; if it could be finite, there might be no non-compact consequence relations on it.)

Finally, reflexivity. 'Reflexive' here is like 'transitive'; it does not have, in its usual application to SET-FMLA or SET-SET consequence relations, its usual relation-theoretic sense. In the usual sense, a relation R on a set S is *reflexive* iff for all $x \in S$, xRx.

For SET-FMLA consequence relations, this cannot apply; these are not relations on a set. What is usually meant by 'reflexivity' in its applications to SET-FMLA relations is either (forgoing abbreviations for the moment): $\{A\} \vdash A$ for all A, or else $\Gamma \cup \{A\} \vdash A$ for all A and Γ. Given monotonicity, these are equivalent to each other.

For SET-SET consequence relations, on the other hand, reflexivity in this sense *can* apply, but almost never *does*. It would require that for every set Γ of formulas, $\Gamma \vdash \Gamma$; but at the very least, the empty set does not entail itself in any familiar setting. There are two usual things one might mean by 'reflexivity' here: that $\Gamma \vdash \Gamma$ for all *singleton* Γ, or all *nonempty* Γ. Given monotonicity, these too are equivalent to each other.

I will not assume reflexivity in what follows, although this turns out not to matter; all the results of the article remain unchanged with such an assumption in place. (To see this, note that all the consequence relations used as counterexamples are reflexive (in the usual senses for consequence relations, not the usual relation-theoretic sense), and that no proof of any claim depends on reflexivity (in any sense).)

That, then, is all the preliminaries.

2.2 *Properties of* SET-FMLA *consequence relations*

The five properties of SET-FMLA consequence relations that generate the present investigation are listed in Table 1. Each is a property that a SET-FMLA consequence relation may or may not exhibit. Each of these is a *closure* property: they are all of the form 'if these things stand in the relation, then those things must also stand in the relation'. They should be understood as universally quantified; for example, $\vdash$ has the property S_{SF} iff whenever $C \vdash A$ and $A \vdash D$, then $C \vdash D$, for all choices of C, A, and D. The properties of SET-FMLA relations that will be considered here are those in Table 1 and arbitrary conjunctions formed from these.

Each allows valid arguments to be *linked* in a specific way; in the antecedent of these properties, the formula A and/or the set Σ of formulas figures among the conclusions of the left conjunct and the premises of the right conjunct, but does not appear in the consequent at all. I will call these properties 'linking' properties.

The abbreviations for the properties are intended to be (at least somewhat) mnemonic without taking up too much space. (I pronounce them 'simple', 'kinda simple', 'suppression', 'finite', and

TABLE 1. Linking properties for SET-FMLA relations

	If	and	then
S_{SF}	$B \vdash A$	$A \vdash C$	$B \vdash C$
$\mathrm{KS}_{\mathrm{SF}}$	$\Gamma \vdash A$	$A \vdash C$	$\Gamma \vdash C$
$\downarrow_{\mathrm{SF}}$	$\vdash A$	$A, \Gamma \vdash C$	$\Gamma \vdash C$
F_{SF}	$\Gamma \vdash A$	$A, \Gamma \vdash C$	$\Gamma \vdash C$
C_{SF}	$\Gamma \vdash A$ for all $A \in \Sigma$	$\Sigma, \Gamma \vdash C$	$\Gamma \vdash C$

TABLE 2. Linking properties for SET-SET relations

	If	and	then
S	$B \vdash A$	$A \vdash C$	$B \vdash C$
KS	$\Gamma \vdash A$	$A \vdash \Delta$	$\Gamma \vdash \Delta$
/↓	$\vdash A$	$A, \Gamma \vdash \Delta$	$\Gamma \vdash \Delta$
↓/	$\Gamma \vdash \Delta, A$	$A \vdash$	$\Gamma \vdash \Delta$
/F	$\Gamma \vdash A$	$A, \Gamma \vdash \Delta$	$\Gamma \vdash \Delta$
F/	$\Gamma \vdash \Delta, A$	$A \vdash \Delta$	$\Gamma \vdash \Delta$
FG	$\Gamma \vdash \Delta, A$	$A, \Gamma \vdash \Delta$	$\Gamma \vdash \Delta$
/C	$\Gamma \vdash A$ for all $A \in \Sigma$	$\Sigma, \Gamma \vdash \Delta$	$\Gamma \vdash \Delta$
C/	$\Gamma \vdash \Delta, \Sigma$	$A \vdash \Delta$ for all $A \in \Sigma$	$\Gamma \vdash \Delta$
/C$^+$	$\Gamma \vdash \Delta, A$ for all $A \in \Sigma$	$\Sigma, \Gamma \vdash \Delta$	$\Gamma \vdash \Delta$
C$^+$/	$\Gamma \vdash \Delta, \Sigma$	$A, \Gamma \vdash \Delta$ for all $A \in \Sigma$	$\Gamma \vdash \Delta$
CG	$\Sigma^+, \Gamma \vdash \Delta, \Sigma^-$ for all partitions $\langle \Sigma^+, \Sigma^- \rangle$ of Σ		$\Gamma \vdash \Delta$

'complete', respectively, but I will stick entirely to the labels given in Table 1 in what follows.) I will discuss this selection in Section 3.

2.3 *Properties of* SET-SET *consequence relations*

Table 2 presents and names twelve properties that a SET-SET consequence relation may or may not exhibit. Each of the properties is again a closure property: 'if these things stand in the relation, then those things must also stand in the relation.' These too should be understood as universally quantified; for example, $\vdash$ has the property KS iff whenever $\Gamma \vdash A$ and $A \vdash \Delta$, then $\Gamma \vdash \Delta$, for all choices of Γ, Δ, and A. The properties of SET-SET relations that will be considered here are those in Table 2 and arbitrary conjunctions formed from these.

As with the SET-FMLA properties, each of these allows valid arguments to be linked in a specific way; in the antecedent of these properties, the formula A and/or the set Σ of formulas figures among the conclusions of the left conjunct and the premises of the right conjunct, but does not appear in the consequent at all. (CG is the only exception to this, as its antecedent does not have left and right conjuncts.) Two of these properties—S and KS—are special cases of transitivity. The others, however, are not.

The abbreviations for the properties are intended to be (at least somewhat) mnemonic without taking up too much space. The properties on this list that have received the most attention are S for 'simple', FG for 'finite generalized', and CG for 'complete generalized'.[4]

A number of the properties in Table 2 are lopsided, focussing in on either the premise or conclusion side of the relation in question. The abbreviations for these properties include a '/'; where the property focusses on the premise side, a letter appears before '/', and where it focusses on the conclusion side, a letter appears after '/'. The 'F' and 'C' are again for 'finite' and 'complete'.

SET-SET consequence relations bring possibilities for symmetry that are absent with SET-FMLA relations. Each property in Table 2 has a *dual* also in the table. Properties P and P' are duals, in the

[4] I take the terms 'simple' and 'generalized' from [26]. Weir's 'simple transitivity' is my S; his 'generalized transitivity' is my FG. (He does not consider CG.)

sense relevant here, iff: for a consequence relation $\vdash$ to have P is for its converse $\dashv$ to have P'. The properties S, KS, FG and CG are all self-dual. For the remaining properties, the names indicate duality; for example, /F and F/ are duals. Also, monotonicity and compactness are both self-dual; a relation $\vdash$ meets each of them iff its converse $\dashv$ does. Noting these symmetries will allow for some of the following proofs about SET-SET relations to get away with only half the work they would otherwise take. For example, once we see that FG implies /F, we can immediately conclude that it implies F/ as well; and once we see that /C$^+$ does not imply C$^+$/, we can immediately conclude that C$^+$/ does not imply /C$^+$ either. I will use this style of reasoning frequently in what follows.

2.4 *Counterparts*

Following [22, Ch. 5], say that a SET-FMLA consequence relation $\vdash$ and a SET-SET consequence relation $\vdash'$ are *counterparts* iff they agree on single-conclusion arguments: $\Gamma \vdash A$ iff $\Gamma \vdash' A$, for every Γ and A. (Note that every SET-FMLA relation has many SET-SET counterparts, while each SET-SET relation has a unique SET-FMLA counterpart.) When a SET-FMLA relation $\vdash$ and a SET-SET relation $\vdash'$ are counterparts, there are some implications that connect Tables 2 and 1.

Extend the notion of counterparts to properties in these tables as follows:

- S and S_{SF} are counterparts.
- KS and KS_{SF} are counterparts.
- $/\!\downarrow$ and $\downarrow_{\text{SF}}$ are counterparts.
- /F and F_{SF} are counterparts.
- /C and C_{SF} are counterparts.
- No other properties are counterparts.

So every property in Table 1 has a counterpart in Table 2, but not vice versa.

FACT 1
A SET-FMLA consequence relation $\vdash$ has a property in Table 1 iff it has a SET-SET counterpart that has the counterpart of that property.

PROOF. RTL is immediate: each instance of the SET-FMLA property in question corresponds directly to an instance of the SET-SET property in question.

LTR: Let $\vdash'_{min}$ be the minimal SET-SET counterpart of $\vdash$, defined as follows: $\Gamma \vdash'_{\text{min}} \Delta$ iff there is a $C \in \Delta$ with $\Gamma \vdash C$. Let P be the SET-FMLA property in question, and let P' be its SET-SET counterpart. Then let $\vdash'$ be the closure of $\vdash'_{min}$ under P'. $\vdash'$ clearly has P'; it remains to show that it is a counterpart of $\vdash$.

Suppose it is not. Then there are Γ, A with $\Gamma \vdash' A$ and $\Gamma \nvdash A$. This must be because there is some chain of applications of property P' to arguments validated by $\vdash'_{min}$ that eventually yields $\Gamma \vdash' A$. If all the arguments in this chain have a singleton set as their conclusion, then a corresponding chain could be run with property P for $\vdash$, and we would have $\Gamma \vdash A$. So at least one argument in this chain has a conclusion that is not a singleton. But each of the five properties that P' could be preserves the property of having a non-singleton conclusion (by inspection), so the chain itself must end in an argument whose conclusion is not a singleton. Contradiction. ■

3 Previous work

Each of the properties considered in Tables 1 and 2 has been considered elsewhere, except possibly for KS_{SF}, which I include to have a counterpart to KS. Indeed, every property I know of that is traveled under the name 'transitivity', and is applicable to SET-FMLA or SET-SET relations, is either in these tables already or else equivalent (given present assumptions) either to one that is or to a conjunction of ones that are.[5] This section provides an overview and some brief discussion.

3.1 The cut rule

In the present setting, F_{SF} is equivalent to: if $\Gamma \vdash A$ and $A, \Gamma' \vdash C$, then $\Gamma, \Gamma' \vdash C$. Similarly, FG is equivalent to the following property: if $\Gamma \vdash \Delta, A$ and $A, \Gamma' \vdash \Delta$, then $\Gamma, \Gamma' \vdash \Delta, \Delta'$. These properties, in turn, are closely connected to [7]'s rules of *cut* in the sequent calculi LJ and LK, respectively. (Just like 'transitivity', 'cut' means many different things in different contexts. Most of them, however, are related to Gentzen's use of 'cut'.)

Cut looms large in many proof-theoretic investigations; F_{SF} and FG, then, have real proof-theoretic import. But they also, at times, have philosophical import. For example, [11, 12] understand FG (as a condition on a particular SET-SET consequence relation) as encoding the following constraint on certain conversational norms: if a certain combination of assertions and denials is within the norms, then for any formula A, either adding an assertion of A to that combination remains within the norms, or else adding a denial of A to that combination remains within the norms. [11, 12] endorse this constraint; [13, 14] dispute it. [24] considers related issues in a SET-FMLA framework, and so deals with F_{SF}.

Other sequent-calculus-like proof systems relate to different properties. For example, the rule called the 'Elimination Rule' in [1, p. 387], 'Display Cut' in [2, p. 203], and just (a 'form' of) 'Cut' in [10, p. 125] is directly connected to KS in the same way that more traditional forms of cut are connected to F_{SF} and FG.

3.2 Suppression

$\downarrow_{\mathrm{SF}}$, $/\!\downarrow$, and $\downarrow\!/$ are properties of *suppression*, in the sense of [16, p. 71], which refers to $/\!\downarrow$ as 'Positive Suppression' and $\downarrow\!/$ as 'Negative Suppression' (and calls $\downarrow_{\mathrm{SF}}$ a 'ridiculous classical property'). See also [16, §2.10] for an in-depth discussion of these properties, including relations to syllogistic and enthymematic argumentation. The authors put forward the 'Anti-Suppression Principle' on p. 146: 'for every statement p there is some statement q such that the consequences of q are a proper subset of the joint consequences of p and q.' The counterparts $\downarrow_{\mathrm{SF}}$ and $/\!\downarrow$, in different frameworks, each say that any theorem is an exception to this principle.

3.3 Bivaluations

One way to present a consequence relation on a language $\mathcal{F}$ is via *bivaluations*: binary partitions $\langle T, F\rangle$ of $\mathcal{F}$. By specifying a set $\mathfrak{M}$ of such partitions, one specifies a consequence relation $\vdash_{\mathfrak{M}}$ in the following ways: for a SET-FMLA consequence, $\Gamma \vdash_{\mathfrak{M}} C$ iff there is no $\langle T, F\rangle \in \mathfrak{M}$ such that

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[5] A property in the area is raised, however, as a form of modus ponens (of all things!) in [20], which operates in the SET-SET framework: if $\vdash A$ and $A \vdash B$, then $\vdash B$. (Thanks to an anonymous referee for bringing this to my attention.) This is a special case of both $/\!\downarrow$ and KS, and is properly weaker than each; the consequence relation later to be called c' exhibits this property but not $/\!\downarrow$, while the consequence relations later to be called a and b exhibit this property but not KS. For these relations, see §5. (It is also a special case of both /F and /C, but we will see in Section 4 that each of these implies both $/\!\downarrow$ and KS, so the property must be properly weaker than these as well.) I would not discuss this property or its relatives further here; properties that the theorems of a consequence relation are closed under are interesting, but must wait for another day.

$\Gamma \subseteq T$ and $C \in F$; for a SET-SET consequence, $\Gamma \vdash_{\mathfrak{M}} \Delta$ iff there is no $\langle T, F \rangle \in \mathfrak{M}$ such that $\Gamma \subseteq T$ and $\Delta \subseteq F$. (Informally, you might think: the argument is valid iff there is no model on which all the premises are true and the conclusion/all the conclusions false.) This way of thinking is stressed in [9, 22], but even where it is not stressed it is often applicable. For example, any way of presenting a consequence relation using models with *designated values* in the usual way fits this mould directly: we can understand each model as partitioning the language into those formulas that receive a designated value and those that do not.

Consequence relations arrived at in this way have certain structural properties: they are *reflexive* (in the senses outlined in Section 2.1), *monotonic*, and have the property C_{SF} or CG, depending on framework. (For proofs, see [22, pp. 16, 30].) As we will shortly see, C_{SF} implies all the other properties in Table 1, and CG implies all the other properties in Table 2. This means that bivaluations will not prove useful in what follows; they obscure the relations between the linking properties under consideration, by forcing them all to hold.[6]

Many monotonic SET-FMLA and SET-SET consequence relations encountered in the wild can be presented in terms of bivaluations, and so exhibit either C_{SF} or CG and thus all the linking properties to be considered here. (Note, however, that [17, p. 83] complains that CG is overstrong, claiming that it requires 'much more than the transitivity of consequence'. In context, Rumfitt is concerned to attack the SET-SET framework entirely; elsewhere he defends a principle he calls 'Cut' that is equivalent (given monotonicity) to C_{SF} [18, p. 42].) It is only in cases where these properties fail that the distinctions explored here are revealed.

3.4 Quantum logic

[6, p. 44] and [4] both consider forms of quantum logic in the SET-SET framework, and attribute to it the conjunction of F/ and /F, which I will call F/F. In quantum logic, distribution of conjunction over disjunction fails; as it happens, there are important connections between distribution and FG.[7] In these authors' settings, quantum logic does not obey FG, which they take to be a default expression of transitivity; F/F is substituted to 'reflect the transitivity of implication' [4, p. 247].

In both cases, the authors restrict their attention to compact relations, for which the conjunction of C/ and /C, which I will call C/C, is equivalent to F/F.[8] (More on compactness presently.) Neither source discusses F/ or /F on their own.

3.5 Neo-classical logic

The 'neo-classical' logic explored in [25, 26], among other places, is another consequence relation that exhibits some of these properties but not others. Weir presents both SET-FMLA and SET-SET versions of neo-classical logic; as [26, p. 100] points out, these obey S_{SF} and S, respectively. In fact they also obey $\mathrm{KS}_{\mathrm{SF}}$ and KS; as we will see, these are stronger. They also obey suppression properties: $\downarrow_{\mathrm{SF}}$ for the SET-FMLA relation and both $\downarrow$/ and /$\downarrow$ for the SET-SET one. However, neither exhibits

[6] However, related techniques from [8, 15] can avoid imposing C_{SF} and CG, as well as every other linking property in these tables.

[7] In particular, the quasi-inequation $x \wedge y \leq z \,\&\, x \leq y \vee z \Rightarrow x \leq z$, which is closely related to FG, holds in a lattice iff the lattice is distributive. This fact is reported by [19, p. 417], who calls it 'well-known'. I have not been able to find a proof in the literature (but see [9, p. 10], Exercise 0.13.7(i), which asks for such a proof), but it is straightforward.

[8] In fact, Dummett (but not Cutland & Gibbins) only considers finite sequents. Note as well that the discussion in [6] in support of F/F, if cogent, in fact supports the full strength of C/C, even for non-compact relations.

any of the other properties in its associated table. [26, fn. 3] claims that S 'should be incorporated in any genuine notion of logical consequence', but does not elaborate; presumably Weir would say the same of $\textsc{s}_{\textsc{sf}}$ for SET-FMLA consequence relations.

3.6 *Cut*$_3$

There is also the property called 'Cut$_3$' in [22, p. 32]. A consequence relation $\vdash$ has Cut$_3$ iff whenever $\Gamma \vdash \Delta, A$ for all $A \in \Sigma_1$, and $B, \Gamma \vdash \Delta$ for all $B \in \Sigma_2$, and $\Sigma_1, \Gamma \vdash \Delta, \Sigma_2$, then $\Gamma \vdash \Delta$. But as is immediately shown there, Cut$_3$ is equivalent to the conjunction of $\textsc{c}^+$/ and /$\textsc{c}^+$; I will later call this conjunction $\textsc{c}^+/\textsc{c}^+$.[9]

[21, p. 37], oddly, calls this property (there defined directly as the conjunction of $\textsc{c}^+$/ and /$\textsc{c}^+$) 'Cut', and takes it to be of some import. In particular, Segerberg points to FG, claims that it is not sufficient when infinite sets of premises and conclusions are considered, and then offers this property as the appropriate replacement, instead of (what one might have expected) CG.[10] He also points out that S, which he calls 'transitivity', is a 'very special case' of this property (p. 38). I know of no other sources that have attended to this property.

4 Implications

This section presents a range of implications that hold among the properties in Tables 1 and 2, and conjunctions formed from these properties.

4.1 *Special cases*

Some implications from one property to another happen in the easiest possible way: when one property covers only certain special cases of another. These implications can be verified directly by inspection, and depend on no special assumptions. For SET-FMLA properties, this gives:

- $\textsc{ks}_{\textsc{sf}}$ implies $\textsc{s}_{\textsc{sf}}$.
- $\textsc{c}_{\textsc{sf}}$ implies $\textsc{f}_{\textsc{sf}}$.

For SET-SET properties, this gives:

- KS implies S.
- /C implies /F.
- C/ implies F/.
- /$\textsc{c}^+$ implies FG.
- $\textsc{c}^+$/ implies FG.

4.2 *Monotonicity*

Other implications are not so direct; these require some appeal to monotonicity. The needed appeals to monotonicity, however, are quite formulaic: when one property's antecedent follows by monotonicity

[9][22, p. 30ff.] considers FG, $\textsc{c}^+$/, /$\textsc{c}^+$, Cut$_3$, and CG; the implications and nonimplications among these properties shown there sit among what is shown in the present article.

[10]As an anonymous referee points out, CG is not mentioned in [21]; nor is [22] cited there.

from another property's antecedent, and they have the same consequent, then the first property implies the second. This gives more implications. For SET-FMLA properties, this gives:

- F_{SF} implies $\mathrm{KS}_{\mathrm{SF}}$.
- F_{SF} implies $\downarrow_{\mathrm{SF}}$.

For SET-SET properties, this gives:

- /F implies KS.
- F/ implies KS.
- /F implies $/\!\downarrow$.
- F/ implies $\downarrow\!/$.
- FG implies /F.
- FG implies F/.
- $/\mathrm{C}^{+}$ implies /C.
- $\mathrm{C}^{+}/$ implies C/.
- CG implies $/\mathrm{C}^{+}$.
- CG implies $\mathrm{C}^{+}/$.

4.3 Compactness

The above-listed implications hold no matter what (within the assumptions that frame this article). For compact relations, there are more implications to take account of among the properties in play; this section records these.

FACT 2
If (SET-SET) $\vdash$ is compact and has FG, then it has CG.

PROOF. See [22, p. 37] for proof. (Their 'cut for formulae' is exactly FG, and their 'cut for sets' is exactly CG.) ∎

FACT 3
If (SET-SET) $\vdash$ is compact and has /F, then it has /C.

PROOF. Suppose $\vdash$ is compact and has /F, that $\Gamma \vdash A$ for all $A \in \Sigma$, and that $\Sigma, \Gamma \vdash \Delta$. Since $\vdash$ is compact, this gives $\Sigma_{\mathrm{fin}}, \Gamma_{\mathrm{fin}} \vdash \Delta_{\mathrm{fin}}$ for some finite $\Sigma_{\mathrm{fin}} \subseteq \Sigma$, $\Gamma_{\mathrm{fin}} \subseteq \Gamma$, and $\Delta_{\mathrm{fin}} \subseteq \Delta$. By monotonicity, $\Sigma_{\mathrm{fin}}, \Gamma \vdash \Delta$. Since $\Sigma_{\mathrm{fin}} \subseteq \Sigma$, we have $\Gamma \vdash A$ for all $A \in \Sigma_{\mathrm{fin}}$. Now, where n is the cardinality of Σ_{fin}, let $\Sigma_{\mathrm{fin}} = \{\sigma_0, \ldots, \sigma_{n-1}\}$, and for $m \leq n$, let $\Sigma^m_{\mathrm{fin}} = \{\sigma_m, \ldots, \sigma_{n-1}\}$. Thus $\Sigma^0_{\mathrm{fin}} = \Sigma_{\mathrm{fin}}$ and $\Sigma^n_{\mathrm{fin}} = \emptyset$.

I claim that for any i from 0 to n (inclusive), $\Sigma^i_{\mathrm{fin}}, \Gamma \vdash \Delta$; when $i = n$, this is $\Gamma \vdash \Delta$, and the proposition follows. This can be shown by induction. The case where $i = 0$ is already shown. So suppose the claim is true for $i < n$; then $\Sigma_i, \Gamma \vdash \Delta$, which is to say $\sigma_i, \Sigma_{i+1}, \Gamma \vdash \Delta$. By assumption, $\Gamma \vdash \sigma_i$; monotonicity gives $\Sigma_{i+1}, \Gamma \vdash \sigma_i$. Now, applying /F, $\Sigma_{i+1}, \Gamma \vdash \Delta$. ∎

FACT 4
If (SET-FMLA) $\vdash$ is compact and has F_{SF}, then it has C_{SF}.

PROOF. Follow the proof of Fact 3, mutatis mutandis. ∎

FACT 5
If (SET-SET) $\vdash$ is compact and has F/, then it has C/.

PROOF. From Fact 3, by duality. ■

4.4 A semilattice of properties

Recall that the goal is to consider not just the properties in Tables 1 and 2, but arbitrary conjunctions of these. Implication among properties forms a semilattice order, with conjunction as the meet.[11] That is, implication is transitive, and the conjunction of two properties is their greatest lower bound wrt the implication order. (In this article, I identify properties that imply each other.) So we can combine the above implications to generate new ones, in the following ways, with P, Q, R any properties (including conjunctions):

- If P implies Q and Q implies R, then P implies R.
- The conjunction of P with Q implies P, and it also implies Q.
- If P implies Q and it also implies R, then P implies the conjunction of Q with R.

4.5 Taking stock

By forming arbitrary conjunctions of the properties in Table 1, we have $2^5 = 32$ distinct ways to specify a SET-FMLA property; and by working with Table 2, we have $2^{12} = 4096$ distinct ways to specify a SET-SET property. However, by identifying properties when they imply each other, we can shrink this list considerably. The implications recorded above are already enough to get us down to 8 SET-FMLA properties and 34 SET-SET properties, or with the assumption of compactness 7 SET-FMLA properties and 18 SET-SET properties.

For each framework, these properties include the empty conjunction of properties, which is trivially exhibited by any consequence relation. I will call this property of SET-FMLA relations $\top_{\text{SF}}$, and of SET-SET relations simply $\top$. All the new properties to consider are given in Tables 3 and 4.

The implications among these properties are recorded in Figures 1 and 2. In these figures, each arrow is an implication already recorded; the double-thickness arrows are implications that we have seen become equivalences in the presence of compactness. (For now, you can ignore the numbers and letters that label the arrows; they'll be meaningful later.)

TABLE 3. Additional SET-FMLA linking properties formed by conjunction

$\top_{\text{SF}}$	The empty conjunction.
$\text{S}{\downarrow}_{\text{SF}}$	S_{SF} and $\downarrow_{\text{SF}}$.
$\text{KS}{\downarrow}_{\text{SF}}$	KS_{SF} and $\downarrow_{\text{SF}}$.

5 Non-implications

So far, only implications have been recorded. So while we know there are *at most* 8 distinct SET-FMLA properties and 34 distinct SET-SET properties in play here, and at most 7 or 18 respectively if compactness is assumed, it's still possible, for all I've said so far, that there are fewer. In fact, however,

[11] For semilattices (and lattices), see [5].

TABLE 4. Additional SET-SET linking properties formed by conjunction

$\top$	The empty conjunction.	$\downarrow/\downarrow$	$/\downarrow$ and $\downarrow/$.
/S/$\downarrow$	S and $/\downarrow$.	$\downarrow$/S/	S and $\downarrow/$.
$\downarrow$/S/$\downarrow$	S and $\downarrow/\downarrow$.	$\downarrow$/KS/$\downarrow$	KS and $\downarrow/\downarrow$.
/KS/$\downarrow$	KS and $/\downarrow$.	$\downarrow$/KS/	KS and $\downarrow/$.
F/$\downarrow$	F/ and $/\downarrow$.	$\downarrow$/F	$\downarrow/$ and /F.
C/$\downarrow$	C/ and $/\downarrow$.	$\downarrow$/C	$\downarrow/$ and /C.
F/F	F/ and /F.	C/C	C/ and /C.
F/C	F/ and /C.	C/F	C/ and /F.
/FG/C	/C and FG.	C/FG/	C/ and FG.
C/FG/C	C/C and FG.	C^+/C^+	$\text{C}^+/$ and $/\text{C}^+$.
C/C^+	C/ and $/\text{C}^+$.	C^+/C	$\text{C}^+/$ and /C.

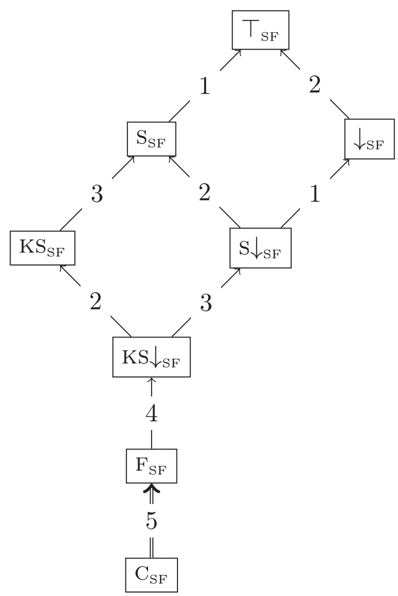

FIGURE 1. Implications for SET-FMLA properties.

there are not; the implications so far recorded exhaust the implications among these properties. This section shows that the remaining potential implications do not hold. In each case, I will show this by counterexample.

5.1 Presenting consequence relations

I will present the consequence relations that serve as counterexamples using a very simple kind of proof system. I work with *sequents*; a SET-FMLA sequent is a pair $\langle \Gamma, C \rangle$, and a SET-SET sequent is a pair $\langle \Gamma, \Delta \rangle$. I will write such pairs with the sequent separator $\rhd$, so $\Gamma \rhd C$ or $\Gamma \rhd \Delta$.

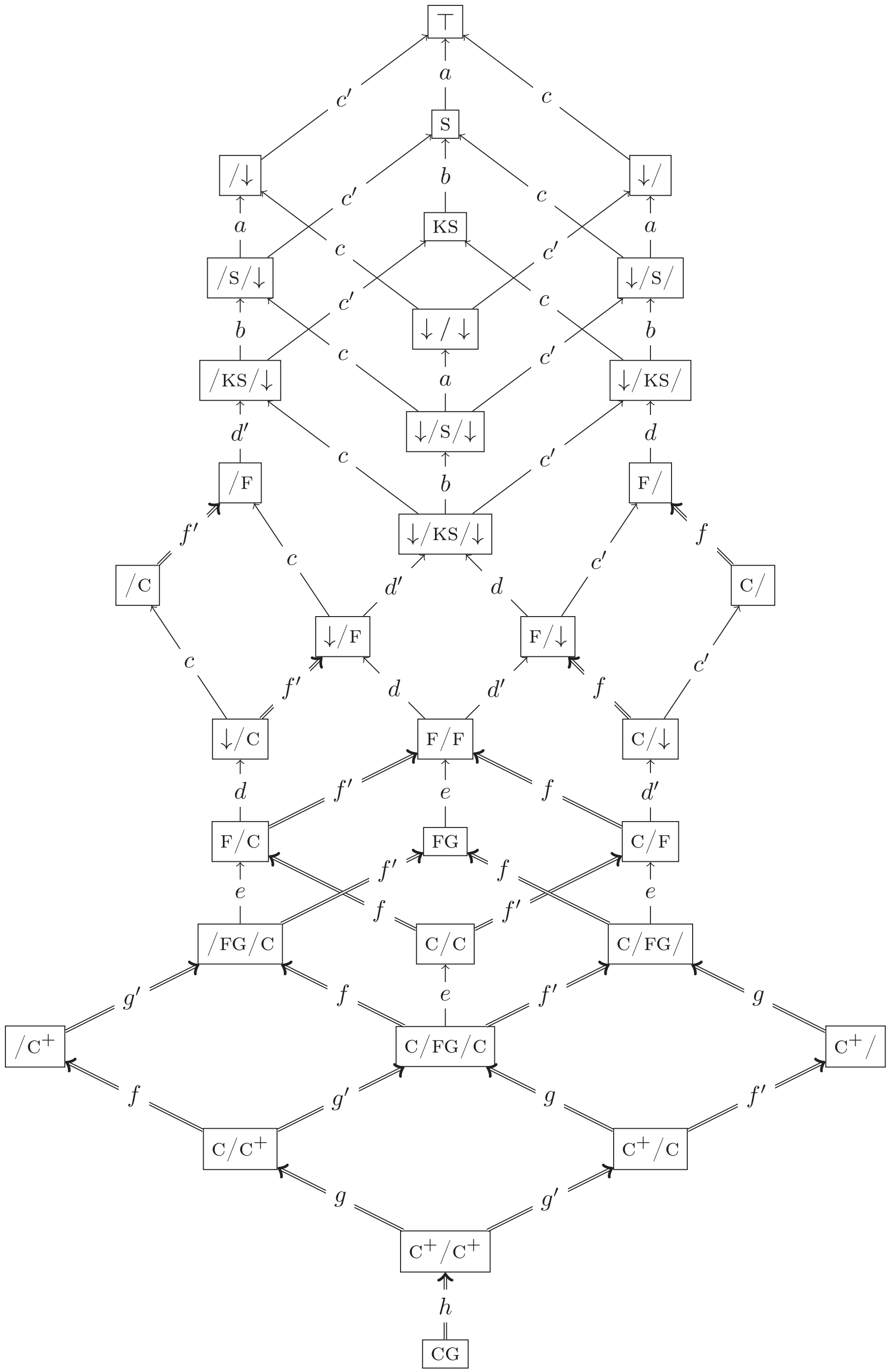

FIGURE 2. Implications for SET-SET properties.

TABLE 5. Five SET-FMLA consequence relations

	$\mathfrak{P}$	Has:	Lacks:
1	$\{p \rhd q;\ q \rhd r\}$	$\downarrow_{\mathrm{SF}}$	S_{SF}
2	$\{\ \rhd p;\ p,q \rhd r\}$	$\mathrm{KS}_{\mathrm{SF}}$	$\downarrow_{\mathrm{SF}}$
3	$\{p,q \rhd r;\ r \rhd s\}$	$\mathrm{S}\!\downarrow_{\mathrm{SF}}$	$\mathrm{KS}_{\mathrm{SF}}$
4	$\{p \rhd q;\ q,r \rhd s\}$	$\mathrm{KS}\!\downarrow_{\mathrm{SF}}$	F_{SF}
5	$\{\Gamma \rhd A : \Gamma$ is infinite or A is not $p\}$	F_{SF}	C_{SF}

A sequent-based proof system involves two components: some set of *initial sequents*, which are simply given as valid, and some *rules* that allow new validities to be generated from old. The proof systems I will draw on here are all quite simple. For each of them, I will specify a set $\mathfrak{P}$ of sequents; the initial sequents of the system are then all those sequents in $\mathfrak{P}$, together with all sequents of the form $A \rhd A$, for any $A \in \mathcal{F}$. There is only a single rule in any of these systems: the rule of *infinitary weakening*. In SET-FMLA systems, this allows us to derive $\Gamma, \Gamma' \rhd C$ from $\Gamma \rhd C$, for any Γ, Γ', C; and in SET-SET systems, this allows us to derive $\Gamma, \Gamma' \rhd \Delta, \Delta'$ from $\Gamma \rhd \Delta$, for any $\Gamma, \Gamma', \Delta, \Delta'$.

So for any set $\mathfrak{P}$ of SET-FMLA sequents, we have a SET-FMLA consequence relation $\vdash_{\mathfrak{P}}$ determined as follows: $\Gamma \vdash_{\mathfrak{P}} C$ iff either 1) $C \in \Gamma$, or 2) there is some $\Gamma' \rhd C \in \mathfrak{P}$ such that $\Gamma' \subseteq \Gamma$. And for any set $\mathfrak{P}$ of SET-SET sequents, we have a SET-SET consequence relation $\vdash_{\mathfrak{P}}$ determined as follows: $\Gamma \vdash_{\mathfrak{P}} \Delta$ iff either 1) $\Gamma \cap \Delta \neq \emptyset$, or 2) there is some $\Gamma' \rhd \Delta' \in \mathfrak{P}$ such that $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$. This approach works at the right level of generality for present purposes: a consequence relation (in either framework) is $\vdash_{\mathfrak{P}}$ for some $\mathfrak{P}$ iff it is monotonic and reflexive. (For the relevant senses of reflexivity, recall the discussion in Section 2.1.) It also gives a tractable way to explore compactness: note that $\vdash_{\mathfrak{P}}$ is compact iff every infinite sequent in $\mathfrak{P}$ has a finite subsequent in $\mathfrak{P}$. (Again, this works in either framework).

5.2 *A menagerie*

Let p, q, r, s be four distinct formulas, and let $\Theta \subseteq \mathcal{F}$ be infinite. Then Table 5 presents five SET-FMLA consequence relations, 1–5, and Table 6 presents eight SET-SET consequence relations, *a*–*h*.[12] For each relation, these tables note two properties: one that the consequence relation has and one that it lacks. These two properties are chosen so that the implications already recorded suffice to settle the situation as regards all the other properties under consideration in the appropriate framework: each other property is either implied by the property the relation has, or else implies the property the relation lacks. (I find this easiest to see by referring to Figure 1 or 2, as appropriate.)

For each of these consequence relations, Table 7 gives a counterexample to the property that it is listed in Table 5 or 6 as lacking; these are easy to check. It is more work to show that each of these relations has the property it is listed as having; proofs of these claims are relegated to the Appendix.

[12][22] gives the relations here called 5 (p. 18) and h (p. 31), for the same purposes: to show that F_{SF} and C_{SF}, and $\mathrm{C}^+/\mathrm{C}^+$ and CG, are distinct. See their Theorems 1.3 (p. 18) and 2.7 (p. 32). They also give a relation (p. 32) similar to the converse of the one here called g, to show that $/\mathrm{C}^+$ and $\mathrm{C}^+/$ are distinct; but they do not consider /C or C/, and their relation does not have C^+/C, as the converse of g does.

TABLE 6. Eight SET-SET consequence relations

	$\mathfrak{P}$	Has:	Lacks:
a	$\{p \rhd q;\ q \rhd r\}$	$\downarrow/\downarrow$	S
b	$\{\Gamma \rhd \Delta : \max(\lvert\Gamma\rvert, \lvert\Delta\rvert) > 2 \text{ and } p \in \Gamma \cup \Delta\}$	$\downarrow$/S/$\downarrow$	KS
c	$\{p \rhd q, r;\ r \rhd\ \}$	/C	$\downarrow$/
d	$\{s \rhd p, q, r;\ p \rhd q, r\}$	$\downarrow$/C	F/
e	$\{q \rhd r, p;\ p, q \rhd r\}$	C/C	FG
f	$\{\Gamma \rhd \Delta : \Delta \text{ is infinite or } \Gamma \cap \Theta \neq \emptyset\}$	/C$^+$	C/
g	$\{\Gamma \rhd \Delta : \Delta \text{ is infinite or } \lvert\Gamma\rvert \geq 2\}$	C/C$^+$	C$^+$/
h	$\{\Gamma \rhd \Delta : \Gamma \cup \Delta \text{ is infinite}\}$	C$^+$/C$^+$	CG

TABLE 7. Counterexamples to properties lacked

1	lacks $\mathrm{S_{SF}}$	$p \vdash q$	$q \vdash r$	$p \nvdash r$
2	lacks $\downarrow_{\mathrm{SF}}$	$\vdash p$	$p, q \vdash r$	$q \nvdash r$
3	lacks $\mathrm{KS_{SF}}$	$p, q \vdash r$	$r \vdash s$	$p, q \nvdash s$
4	lacks $\mathrm{F_{SF}}$	$p \vdash q$	$q, r \vdash s$	$p, r \nvdash s$
5	lacks $\mathrm{C_{SF}}$	$\vdash A$ for all $A \in \mathcal{F} \setminus \{p\}$	$\mathcal{F} \setminus \{p\} \vdash p$	$\nvdash p$
a	lacks S	$p \vdash q$	$q \vdash r$	$p \nvdash r$
b	lacks KS	$q, r \vdash p$	$p \vdash s, t$	$q, r \nvdash s, t$
c	lacks $\downarrow$/	$p \vdash q, r$	$r \vdash$	$p \nvdash q$
d	lacks F/	$s \vdash p, q, r$	$p \vdash q, r$	$s \nvdash q, r$
e	lacks FG	$q \vdash r, p$	$p, q \vdash r$	$q \nvdash r$
f	lacks C/	$\vdash \Theta$	$A \vdash$ for all $A \in \Theta$	$\nvdash$
g	lacks C$^+$/	$p \vdash \mathcal{F} \setminus \{p\}$	$p, A \vdash$ for all $A \in \mathcal{F} \setminus \{p\}$	$p \nvdash$
h	lacks CG	$\mathcal{F}^+ \vdash \mathcal{F}^-$ for every partition $\langle \mathcal{F}^+, \mathcal{F}^- \rangle$ of $\mathcal{F}$		$\nvdash$

5.3 *Full characterization*

Let us take stock. Each of Figures 1 and 2 records implications between properties remarked in Section 4. Where those implications were shown to become equivalences in the presence of compactness, that too is recorded. Now, each of those implications has been shown to be one-way, by direct counterexample. None of them collapses to an equivalence without compactness.

Moreover, where equivalence in the presence of compactness was not already shown, the counterexample given is in fact compact. That is, relations 1–4 and *a*–*e* are all compact, and every single-thickness line in Figures 1 and 2 is labelled with one of these.[13] So, with the assumption of compactness added, no implication collapses to an equivalence except for those already noted in Section 4.

Now, suppose there were some implication between some of the properties in one of these figures that was not already recorded in the figure; say property P implies property Q. Because of how the figures were constructed, the conjunction of P with Q appears in the figure, and the implication from

[13]For all of these but b, compactness is immediate: $\mathfrak{P}$ only includes finite sequents. For b, it is quick to see that every infinite sequent in $\mathfrak{P}$ has a finite subsequent in $\mathfrak{P}$.

this conjunction to Q is recorded. But if P implies Q, then this implication is really an equivalence. That is, because these properties are closed under conjunction, and because all implications from conjunctions to their conjuncts are indicated, showing that no indicated implication collapses to equivalence suffices to show that there are no implications beyond those indicated.

Figures 1 and 2 thus suffice to completely characterize the implications and non-implications among arbitrary conjunctions of properties drawn from Tables 1 and 2, both with and without the assumption of compactness.

6 Conclusion

'Transitivity', as applied to consequence relations, can conceal more than it reveals. When someone says a consequence relation is 'transitive', then, it is worth finding out just what is meant. It is almost never the case that they mean that it is *transitive*, in the usual relation-theoretic sense. But then what do they mean?

This article has explored some possible answers, for both SET-FMLA and SET-SET consequence relations. It's a safe bet that nobody means $\top_{\mathrm{SF}}$ or $\top$ by 'transitivity', but the remaining properties in question are all possible ways to fill in the idea. When *we* call consequence relations 'transitive', then, it behooves us to make clear exactly what we are saying; there is no single thing we must obviously mean.

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Appendix

A Proofs of claims in Section 5.2

Here, I prove that each of the consequence relations 1–5 and *a*–*h* has the property it is claimed to have in Table 5 or 6. These proofs are largely tedious rather than exciting; this section is accordingly probably best treated as reference material rather than browsed.[14]

1 has $\downarrow_{\mathrm{SF}}$

Proof. No argument with an empty set of premises is valid, so there is no opportunity for a counterexample. ■

2 has $\mathrm{KS}_{\mathrm{SF}}$

Proof. Suppose $\Gamma \vdash_2 A$ and $A \vdash_2 C$. Since $A \vdash_2 C$, it must be that either A and C are the same sentence or C is p. In the first case, $\Gamma \vdash_2 C$ by assumption; in the second, by monotonicity from $\vdash_2 p$. ■

[14] But who am I to tell you what to do with the article?

3 has $\text{S}{\downarrow}_{\text{SF}}$

PROOF. $\downarrow_{\text{SF}}$ is immediate, since no argument with an empty set of premises is valid. For S_{SF}, suppose $B \vdash_3 A$ and $A \vdash_3 C$. There are three ways this can happen:

- B is r and A is s. In this case, C must be s, so $B \vdash_3 C$.
- A is r and C is s. In this case, B must be r, so $B \vdash_3 C$.
- Otherwise, B must be A and A must be C. So $B \vdash_3 C$.

■

4 has $\text{KS}{\downarrow}_{\text{SF}}$

PROOF. Again, $\downarrow_{\text{SF}}$ is immediate, since no argument with an empty set of premises is valid. For KS_{SF}, suppose $\Gamma \vdash_4 A$ and $A \vdash_4 C$. If A is C, then $\Gamma \vdash_4 C$ and we're done. So suppose A is not C; then it must be that A is p and C is q. But then $\Gamma \vdash_4 p$, and so $p \in \Gamma$. Since $p \vdash_4 q$, this gives $\Gamma \vdash_4 C$ by monotonicity. ■

5 has F_{SF}

PROOF. ([22, p. 18] states but does not prove this result.) Suppose $\Gamma \vdash_5 A$ and $A, \Gamma \vdash_5 C$. If Γ is infinite, $\Gamma \vdash_5 C$ and we're done; so suppose Γ is finite. If C is not p, then $\Gamma \vdash_5 C$ and we're done; so suppose C is p. Finally, if $p \in \Gamma$, then $\Gamma \vdash_5 C$ (since C is p) and we're done; so suppose $p \notin \Gamma$. These suppositions leave us with $A, \Gamma \vdash_5 p$ with Γ finite and $p \notin \Gamma$; so it must be that A is p. But then $\Gamma \vdash_5 p$ by assumption, and since C is p, this gives $\Gamma \vdash_5 C$. ■

a has $\downarrow / \downarrow$

PROOF. No argument with an empty set of premises or an empty set of conclusions is valid, so there is no opportunity for a counterexample to either $/\downarrow$ or $\downarrow/$. ■

b has $\downarrow/\text{S}/\downarrow$

PROOF. For $\downarrow / \downarrow$: no argument with an empty set of premises and a singleton set of conclusions is valid; nor is any argument with a singleton set of premises and an empty set of conclusions.

For S: suppose $B \vdash_b A$ and $A \vdash_b C$. Then it must be that B is A and that A is C. So $B \vdash_b C$. ■

c has $/\text{C}$

PROOF. Suppose that $\Gamma \vdash_c A$ for all $A \in \Sigma$ and that $\Sigma, \Gamma \vdash_c C$. If $r \in \Gamma$, then $\Gamma \vdash_c C$ and we're done, so suppose $r \notin \Gamma$. Then it must be that $A \in \Gamma$ for all $A \in \Sigma$; that is, that $\Sigma \subseteq \Gamma$. But then $\Sigma \cup \Gamma = \Gamma$, and so since $\Sigma, \Gamma \vdash_c C$ we have $\Gamma \vdash_c C$. ■

d has $\downarrow/\text{C}$

PROOF. $\downarrow/$ is immediate, since no argument with an empty set of conclusions is valid. For $/\text{C}$, suppose that $\Gamma \vdash_d A$ for all $A \in \Sigma$ and that $\Sigma, \Gamma \vdash_d C$. Then it must be that $A \in \Gamma$ for all $A \in \Sigma$; that is, that $\Sigma \subseteq \Gamma$. But then $\Sigma \cup \Gamma = \Gamma$, and so since $\Sigma, \Gamma \vdash_d C$ we have $\Gamma \vdash_d C$. ■

e has C/C

Proof. /C first. Suppose there are Γ, Δ, Σ such that $\Gamma \vdash_e A$ for every $A \in \Sigma$ and $\Sigma, \Gamma \vdash_e \Delta$. If $\Sigma \subseteq \Gamma$ then $\Gamma \vdash_e \Delta$ and we're done; so take any $B \in \Sigma$ such that $B \notin \Gamma$. Since $\Gamma \vdash_e B$ and $B \notin \Gamma$, it must be that $p, q \in \Gamma$ and B is r. That is, $\Sigma \cup \Gamma = \{r\} \cup \Gamma$. Thus, since $\Sigma, \Gamma \vdash_e \Delta$, it must be that either $\Gamma \cap \Delta$ is non-empty or $r \in \Delta$; but either way $\Gamma \vdash_e \Delta$.

For C/, the argument is dual, reversing the roles of q and r.

■

f has /C$^+$

Proof. Suppose $\Gamma \vdash_f \Delta, A$ for each $A \in \Sigma$, and that $\Sigma, \Gamma \vdash_f \Delta$. For each $A \in \Sigma$ we can have $\Gamma \vdash_f \Delta, A$ in only four ways:

(1) $A \in \Gamma$,
(2) $\Gamma \cap \Delta \neq \emptyset$,
(3) $\Gamma \cap \Theta \neq \emptyset$, or
(4) Δ is infinite.

In cases 2–4, $\Gamma \vdash_f \Delta$ directly. All that remains is the case where $A \in \Gamma$ for every $A \in \Sigma$; that is, where $\Sigma \subseteq \Gamma$. But we have $\Sigma, \Gamma \vdash_f \Delta$, and so in this case too $\Gamma \vdash_f \Delta$. ■

g has C/C$^+$

Proof. First, C/: if $A \vdash_g \Delta$ for each $A \in \Sigma$, then either Δ is infinite, in which case $\Gamma \vdash_g \Delta$ directly, or else $\Sigma \subseteq \Delta$; the only valid arguments with finitely many conclusions and a single premise are those where the premise is among the conclusions. But if $\Sigma \subseteq \Delta$, then if $\Gamma \vdash_g \Delta, \Sigma$, this is already $\Gamma \vdash_g \Delta$.

Second, /C$^+$: suppose that $\Gamma \vdash_g \Delta, A$ for all $A \in \Sigma$ and $\Sigma, \Gamma \vdash_g \Delta$. If Δ is infinite, then $\Gamma \vdash_g \Delta$ and we're done; so suppose Δ is finite. It must be, then, that $\Sigma \cup \Gamma$ has at least two members. If Γ itself has at least two members, then $\Gamma \vdash_g \Delta$ and we're done, so suppose it does not. There must be at least one formula, then, in Σ but not in Γ; call it B. By assumption, $\Gamma \vdash_g \Delta, B$. But since Γ does not have at least two members and Δ is not infinite, this means that $\Gamma \cap (\Delta \cup \{B\})$ is non-empty; and since $B \notin \Gamma$, it must be that $\Gamma \cap \Delta$ is non-empty. So $\Gamma \vdash_g \Delta$.

■

h has C$^+$/C$^+$

Proof. ([22, p. 33] states but does not prove this result.)

For /C$^+$: suppose $\Gamma \vdash_h \Delta, A$ for all $A \in \Sigma$, and that $\Sigma, \Gamma \vdash_h \Delta$, to show $\Gamma \vdash_h \Delta$. For each $A \in \Sigma$, we can have $\Gamma \vdash \Delta, A$ in only three ways:

(1) $A \in \Gamma$,
(2) $\Gamma \cap \Delta \neq \emptyset$, or
(3) $\Gamma \cup \Delta \cup \{A\}$ is infinite.

In case 3, $\Gamma \cup \Delta$ is also infinite, and so $\Gamma \vdash_h \Delta$. In case 2, $\Gamma \vdash_h \Delta$ as well. So we need only consider the case where $A \in \Gamma$ for each $A \in \Sigma$; that is, where $\Sigma \subseteq \Gamma$. But we have $\Sigma, \Gamma \vdash_h \Delta$; and so in this case too $\Gamma \vdash_h \Delta$.

$\vdash_h$ is self-dual and has /C$^+$, so it has C$^+$/ as well; thus, it has C$^+$/C$^+$. ■