Uniqueness without reflexivity or transitivity

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Uniqueness

Two families

Without id and cut

Uniqueness

What is uniqueness?

The key question:

When does a set of rules uniquely characterize a connective?

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A rule is a schema of the form

$$\frac{S_1 \quad S_2 \quad \ldots \quad S_n}{S}$$

where $S, S_1, S_2, \ldots, S_n$ are schematic sequents.

Example:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \to B \vdash \Delta, \Delta'}$$

Do these rules pin down a unique connective $\to$?
(From a multiple-conclusion intuitionist calculus)

Uniqueness

Why it matters

Some inferentialists hold that

the meaning of a connective is given by the rules governing its use.

But then at least some collections of rules must be able to give a particular meaning.

Belnap (1962):

"It seems rather odd to say we have defined *plonk* unless we can show that *A*-*plonk*-*B* is a function of *A* and *B*, *i.e.* given *A* and *B*, there is only one proposition *A*-*plonk*-*B*."

What is it to be 'only one proposition'?

Uniqueness also matters for combining logics.

Suppose:

- Rules R suffice for unique characterization
- $\star$ obeys exactly rules R in logic L1
- $\dagger$ obeys rules R plus S in logic L2

There will be trouble combining L1 and L2;
$\star$ and $\dagger$ must be the same connective in the combined logic,
but they cannot be.

Uniqueness has been made precise in multiple ways.

These fall into two broad families:
the sub family
and the id family.

Two families

The sub family

Belnap (1962) connects uniqueness to 'inferential role',
by which he understands:

$n$ary $\star$ and $\dagger$ have the same inferential role:

$$\frac{\Gamma, \star(A_1, \ldots, A_n) \vdash \Delta}{\Gamma, \dagger(A_1, \ldots, A_n) \vdash \Delta} \qquad \frac{\Gamma \vdash \star(A_1, \ldots, A_n), \Delta}{\Gamma \vdash \dagger(A_1, \ldots, A_n), \Delta}$$

For Belnap, rules are uniquely characterizing iff:
giving the same rules to $\star$ and $\dagger$
leaves all four of these rules admissible.

Belnap's condition is an instance of the sub family:
it is about when one connective can be substituted for another.

Two possible variations:

- require derivability, rather than just admissibility
- allow substitution in embedded uses, rather than just main

So the sub family has four members;
all are nonequivalent,
and Belnap's is the weakest.

Two families

The id family

Humberstone requires a very different condition for uniqueness:

Humberstone:

$C(\star(A_1, \ldots, A_n)) \dashv\vdash C(\dagger(A_1, \ldots, A_n))$
for any formula context $C(\ )$.

For Humberstone, rules are uniquely characterizing iff:
giving the same rule to $\star$ and $\dagger$
results in validating these arguments.

Humberstone's condition is an instance of the id family:
it is about deriving variations on identity sequents:

Id: $$\frac{}{A \vdash A}$$

Again, we can allow embedding or restrict to main occurrences.
(Humberstone, unlike Belnap, allows embedding.)

There is no difference between admissibility and derivability for individual arguments.

So the id family has two members; Humberstone's is the stronger.

Two families

When they are equivalent

The sub family and the id family are clearly not the same.

But they are related;
in many cases members of these families turn out equivalent.

Cut:

$$\text{Cut:} \quad \frac{\Gamma \vdash A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

If cut is admissible/derivable
and some member of the id family holds,
then the corresponding member of the sub family holds.

One of four needed derivations:

$$\text{Cut:} \quad \frac{\Gamma \vdash C(\star(A_1, \ldots, A_n)), \Delta \qquad C(\star(A_1, \ldots, A_n)) \vdash C(\dagger(A_1, \ldots, A_n))}{\Gamma \vdash C(\dagger(A_1, \ldots, A_n)), \Delta}$$

Id:

$$\text{Id:} \quad \frac{}{A \vdash A}$$

If id holds
and some member of the sub family holds,
then the corresponding member of the id family holds.

$$\frac{\star(A_1, \ldots, A_n) \vdash \star(A_1, \ldots, A_n)}{\star(A_1, \ldots, A_n) \vdash \dagger(A_1, \ldots, A_n)}$$

Overall, in the presence of id and cut, we have ID iff SUB,

so long as:

- the admissible/derivable parameter in SUB matches the status of cut
- ID and SUB match on whether they allow embedding

Corollary:

In the presence of id and derivable cut, sub rules are derivable iff admissible.

Without id and cut

Id vs sub

We might be interested, however, in logics without id and cut.

In these cases, the sub family and the id family can diverge.

This divergence can tell us about the more usual cases as well; exactly what is important about these conditions?

For inferentialism:
when have we defined a single connective?

For combining logics:
When does collapse threaten?

Suppose:
$\star(A_1, \ldots, A_n) \vdash \dagger(A_1, \ldots, A_n)$, but
$\star(A_1, \ldots, A_n) \nvdash \star(A_1, \ldots, A_n)$.

This doesn't seem like the same connective at all.

For these uses, the sub family gets at what we're after.

Within the sub family:

Embeddings or main formula only?
Derivable or only admissible?

For inferentialism:
admissibility needs ‘that’s all’ clause in definitions,
while derivability can do without.

For combining:
only derivability causes trouble;
admissibility allows combination without issue.

- When do rules specify a unique connective?
- This matters for inferentialism and combining logics.
- The sub family and the id family give two strategies for understanding this.
- They are equivalent in the presence of id and cut.
- Without id and cut, the sub family—and not the id family—gets at what matters.