Recapture, ambiguity, and conflation

David Ripley

Monash and Unimelb
https://davewripley.rocks

Life without contraction

Semantic paradoxes (eg liar, curry) are one motivation for many kinds of nonclassical logic.

The focus today:
multiplicative-additive linear logic (MALL)
and multiplicative-additive affine logic (MAAL) in this role.

MALL and MAAL have unary $\neg$ and binary and nullary connectives:

	$\wedge$	$\vee$	$\rightarrow$	T	F
$\times$ive:	$\otimes$	$⅋$	$\multimap$	t	f
$+$ive:	$\sqcap$	$\sqcup$	$\sqsupset$	$\top$	$\bot$

Call this language $\mathcal{L}_{\mathrm{L}}$,

and call the classical language (with $\neg$, $\wedge$, $\vee$, $\rightarrow$, T, F) $\mathcal{L}_{\mathrm{C}}$.

At a given arity,
the +ives are interdefinable via negation,
as are the ×ives.

$$
\begin{array}{lll}
A \sqcap B & = & \neg(A \sqsupset \neg B) \\
A \sqcup B & = & \neg(\neg A \sqcap \neg B) \\
A \sqsupset B & = & \neg A \sqcup B \\
\\
\top & = & \neg\bot \\
\bot & = & \neg\top
\end{array}
\qquad
\begin{array}{lll}
A \otimes B & = & \neg(A \multimap \neg B) \\
A \parr B & = & \neg(\neg A \otimes \neg B) \\
A \multimap B & = & \neg A \parr B \\
\\
\mathsf{t} & = & \neg\mathsf{f} \\
\mathsf{f} & = & \neg\mathsf{t}
\end{array}
$$

So for $\mathcal{L}_{\mathrm{L}}$ I'll use just $\otimes, \sqcap, \mathsf{t}, \top$
and for $\mathcal{L}_{\mathrm{C}}$ just $\wedge, \mathrm{T}$

$$\sqcap L: \quad \frac{[A/B, \Gamma \succ \Delta]}{[A \sqcap B, \Gamma \succ \Delta]} \qquad \sqcap R: \quad \frac{[\Gamma \succ \Delta, A] \qquad [\Gamma \succ \Delta, B]}{[\Gamma \succ \Delta, A \sqcap B]}$$

$$\otimes L: \quad \frac{[A, B, \Gamma \succ \Delta]}{[A \otimes B, \Gamma \succ \Delta]} \qquad \otimes R: \quad \frac{[\Gamma \succ \Delta, A] \qquad [\Gamma' \succ \Delta', B]}{[\Gamma, \Gamma' \succ \Delta, \Delta', A \otimes B]}$$

$\top$R: $$\frac{}{[\Gamma \succ \Delta, \top]}$$

tL: $$\frac{[\Gamma \succ \Delta]}{[t, \Gamma \succ \Delta]}$$ tR: $$\frac{}{[\ \succ t]}$$

Structurally, MALL gives just Id and Cut.

(Remember we're using multisets!)

Multiplicative-additive affine logic MAAL
adds Dilution to MALL:

$$\text{D:}\quad \frac{[\Gamma \succ \Delta]}{[\Sigma, \Gamma \succ \Delta, \Theta]}$$

(All sequents are finite, so this is the same as diluting one formula at a time)

In MAAL:

$$A \otimes B \vdash A \sqcap B$$

$$A \sqcup B \vdash A \parr B$$

$$A \sqsupset B \vdash A \multimap B$$

but not vice versa.

MALL gives none of these.

Both MALL and MAAL avoid paradox-driven trouble:
even in the presence of transparent truth and paradoxical sentences
they remain nontrivial.

Classical recapture

Suppose that some nonclassical logic gives the right story about truth.

(Whatever that means.)

Suppose too that classical logic works fine for other purposes.

Then there is an explanatory gap to be filled.

If classical logic is wrong,
why does it work so well so much of the time?

Candidate answers often involve interesting relations between some favoured logic and classical logic.

Examples

Let $\Sigma?$ be $\{p \vee \neg p \mid p \in At(\Sigma)\}$.
Then $\Gamma \vdash_{CL} \Delta$ iff $\Gamma, \Gamma?, \Delta? \vdash_{K3} \Delta$.

Let $\Sigma!$ be $\{p \wedge \neg p \mid p \in At(\Sigma)\}$.
Then $\Gamma \vdash_{CL} \Delta$ iff $\Gamma \vdash_{LP} \Gamma!, \Delta!, \Delta$.

So a K3 partisan might explain classical success as involving suppressed premises, and an LP partisan suppressed conclusions.

Ambiguity

In a range of work, Paoli and others have dealt with an alleged ambiguity in certain connectives.

The ambiguity is the $+$ive / $\times$ive one we've met.

Two claims:
— The natural language connectives ('and', 'or', etc) are ambiguous in this way
— The $\mathcal{L}_{\mathrm{C}}$ connectives ($\wedge$, $\vee$, etc) are ambiguous in this way

In a range of work, Paoli and others have dealt with an alleged ambiguity in certain connectives.

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"Implicational paradoxes..." (Paoli 2007):

"Classical logic is not so much wrong—if by this word we mean that it ascribes disputable properties to the logical constants it deals with—as ambiguous: its connectives are ill-defined inasmuch as they have multiple meanings."

"Logical consequence and the paradoxes" (Mares & Paoli 2014):

"There is no need to give up any compelling inferential principle of classical logic—only to recognize that bad things can happen when principles holding of different connectives are used, in the course of a derivation, as holding of the same ambiguous connective..."

The real story is meant to be given by MALL:
classical logic, even where it goes beyond MALL,
is not wrong but merely expressing correct things ambiguously.

What are the facts meant to support this picture?

Mares & Paoli point to the Ono translations $o^{\pm} : \mathcal{L}_{\mathrm{C}} \to \mathcal{L}_{\mathrm{L}}$,
based on the Grišin translations $\gamma^{\pm} : \mathcal{L}_{\mathrm{C}} \to \mathcal{L}_{\mathrm{L}}$.

A	$\gamma^+(A)$	$\gamma^-(A)$	$o^+(A)$	$o^-(A)$
p	p	p	$\mathsf{f} \sqcup p$	$\mathsf{t} \sqcap p$
T	$\top$	t	same	
$\neg B$	$\neg\gamma^-(B)$	$\neg\gamma^+(B)$	same	
$B \wedge C$	$\gamma^+(B) \sqcap \gamma^+(C)$	$\gamma^-(B) \otimes \gamma^-(C)$	same	

Facts (Grišin, Ono):

$$\Gamma \vdash_{\mathrm{CL}} \Delta \text{ iff } \gamma^{-}(\Gamma) \vdash_{\mathrm{MAAL}} \gamma^{+}(\Delta)$$

$$\Gamma \vdash_{\mathrm{CL}} \Delta \text{ iff } o^{-}(\Gamma) \vdash_{\mathrm{MALL}} o^{+}(\Delta)$$

The difference in atoms is exactly to ensure dilution:

Already in MALL, dilution is inductive:
if we can dilute with all the atomic sentences in A,
we can dilute with A itself.

$o^{\pm}$ gives $\mathsf{t} \sqcap p$ in negative positions and $\mathsf{f} \sqcup p$ in positive:

$$\sqcap\text{L}: \dfrac{\text{tL}: \dfrac{[\Gamma \succ \Delta]}{[\mathsf{t}, \Gamma \succ \Delta]}}{[\mathsf{t} \sqcap p, \Gamma \succ \Delta]} \qquad \sqcup\text{R}: \dfrac{\text{fR}: \dfrac{[\Gamma \succ \Delta]}{[\Gamma \succ \Delta, \mathsf{f}]}}{[\Gamma \succ \Delta, \mathsf{f} \sqcup p]}$$

Problem:

These technical facts don't fit the 'ambiguity' story, for two reasons:

- one where the difference between $o^{\pm}$ and $\gamma^{\pm}$ matters
- one that hits both $o^{\pm}$ and $\gamma^{\pm}$ equally

First, the Ono translations require ‘disambiguating’ atomic sentences, but this is unmotivated and possibly vicious.

The alleged ambiguity is in the classical connectives; we’ve been given no reason to suspect ambiguity in the atoms.

Also, p itself occurs in both $\mathrm{t} \sqcap p$ and $\mathrm{f} \sqcup p$; are these also ambiguous?

Mares & Paoli offer a fallback:

Mares & Paoli 2014:

"[I]f we confine ourselves to the classical tautologies that play a role in the known versions of the paradoxes, you do not need to replace propositional variables in order to get to a theorem of [MALL]."

That is, they suggest using $\gamma^{\pm}$ over MALL, rather than $o^{\pm}$.

But this simply does not work
to discharge the explanatory debt.

For example, $p \wedge q \vdash_{\mathrm{CL}} p$,
but $\gamma^{-}(p \wedge q) = p \otimes q \nvdash_{\mathrm{MALL}} p = \gamma^{+}(p)$.

Lots of ordinary classical principles turn out wrong after all,
and so there is no explanation for classical success.

Maybe, though, just explaining the success of “the classical tautologies that play a role in the known versions of the paradoxes” is enough?

This would abandon the full-scale explanatory project,
but perhaps handle an important piece of it.

ϕ	$\gamma^+(\phi)$	MALL theorem?
$p \vee \neg p$	$p \parr \neg p$	✓
$\neg(p \wedge \neg p)$	$\neg(p \otimes \neg p)$	✓

ϕ	$\gamma^+(\phi)$	MALL theorem?
$p \vee \neg p$	$p \parr \neg p$	✓
$\neg(p \wedge \neg p)$	$\neg(p \otimes \neg p)$	✓
$(p \to (p \to q)) \to (p \to q)$	$(p \sqsupset (p \sqsupset q)) \multimap (p \multimap q)$	×
$(p \wedge (p \to q)) \to q$	$(p \otimes (p \sqsupset q)) \multimap q$	×

Summing up the first problem:

The story doesn't motivate Ono's translation, only (at most) Grišin's.

This reaches CL if we start from MAAL, but not from MALL;
and even Mares & Paoli's weaker claim about MALL/γ fails.

There is room here for an advocate of MAAL, but not of MALL.

There is also a separate problem,
this one for MALL/o and MAAL/γ alike:

The mere existence of a true disambiguation
does not make a pronouncement true.

So just the existence of $\gamma^{\pm}$ is not enough to justify
even those classical theorems that are in its image.

When is an ambiguous pronouncement nonetheless right?

I don't know: some combination of context, speaker intent, information available to hearer(s), . . . ?

However ambiguity is resolved, though, it's not
a matter of premises, conclusions,
positive and negative occurrences, and so forth.

It's not like 'I saw a bat'
is about animals when it's a premise
and sporting equipment when it's a conclusion.

The philosophical story we are given
does not fit the logical facts appealed to.

Conflation

We conflate things when we treat them as one.

Among the things we can conflate are propositions:
consider $\forall\exists$ vs $\exists\forall$ scope difficulties,
or $A \to \Box B$ vs $\Box(A \to B)$.

Perhaps classical connectives express the conflations of their linear counterparts.

$A \wedge B$	conflates	$A \otimes B$	with	$A \sqcap B$,
$A \vee B$	conflates	$A \parr B$	with	$A \sqcup B$,
$A \to B$	conflates	$A \multimap B$	with	$A \sqsupset B$,
T	conflates	t	with	$\top$,
F	conflates	f	with	$\bot$.

Take the translation $\beta : \mathcal{L}_{\mathrm{L}} \to \mathcal{L}_{\mathrm{C}}$:

A	$\beta(A)$
p	p
$\top$	T
t	T

A	$\beta(A)$
$\neg B$	$\neg\beta(B)$
$B \otimes C$	$\beta(B) \wedge \beta(C)$
$B \sqcap C$	$\beta(B) \wedge \beta(C)$

Given a formula in $\mathcal{L}_{\mathrm{L}}$, this gives us the corresponding $\mathcal{L}_{\mathrm{C}}$ formula ignoring the $+$ive / $\times$ive distinction.

I've elsewhere (2017, 2018) defended:

Conflation by blurring:

If $\beta : \mathcal{L}_1 \to \mathcal{L}_2$ registers the ways $\mathcal{L}_1$ is conflated in $\mathcal{L}_2$,
and $\vdash$ captures $\mathcal{L}_1$ validity,
then $\vdash^\beta$ captures $\mathcal{L}_2$ validity, where

$\Gamma \vdash^\beta \Delta$ iff there are Γ', Δ' such that:

$$\Gamma' \vdash \Delta' \text{ and } \beta(\Gamma') = \Gamma \text{ and } \beta(\Delta') = \Delta$$

(For sets, not multisets; but let's try the multiset version.)

It's handy to have proof systems for $\mathcal{L}_{\mathrm{C}}$
patterned after MALL and MAAL, using γ and β:

- FULL includes all the rules that come from MAAL via β;
- G3ISH includes all the rules that γ takes to rules of MAAL;
- NOD includes all the rules that come from MALL via β;
- NOD^{-} includes all the rules that γ takes to rules of MALL

(See handout.)

FULL and G3ISH are sound and complete for CL (therefore admit cut), and even have the same derivable rules as each other.

NOD and NOD$^-$ differ, and are both nonclassical:

$$p, q \nvdash_{\text{NOD}} q \text{ and } p \wedge q \vdash_{\text{NOD}} q$$

$$p, q \nvdash_{\text{NOD}^-} q \text{ and } p \wedge q \nvdash_{\text{NOD}^-} q$$

These systems (plus noting that $\beta \circ \gamma^{\pm}$ is the identity on $\mathcal{L}_{\mathrm{C}}$) make for quick analogs of the Grišin result:

Four results

- (Grišin) $\Gamma \vdash_{\mathrm{G3ISH}} \Delta$ iff $\gamma^{-}(\Gamma) \vdash_{\mathrm{MAAL}} \gamma^{+}(\Delta)$
- $\Gamma \vdash_{\mathrm{NOD}^{-}} \Delta$ iff $\gamma^{-}(\Gamma) \vdash_{\mathrm{MALL}} \gamma^{+}(\Delta)$
- $\Gamma \vdash_{\mathrm{FULL}} \Delta$ iff there are Γ', Δ' such that:
  $\Gamma' \vdash_{\mathrm{MAAL}} \Delta'$ and $\beta(\Gamma') = \Gamma$ and $\beta(\Delta') = \Delta$
- $\Gamma \vdash_{\mathrm{NOD}} \Delta$ iff there are Γ', Δ' such that:
  $\Gamma' \vdash_{\mathrm{MALL}} \Delta'$ and $\beta(\Gamma') = \Gamma$ and $\beta(\Delta') = \Delta$

This gives a new kind of recapture for MAAL:

CL is indeed just what we get from MAAL by conflating $+$ive / $\times$ive.

So if MAAL is the right logic, we can fully explain the success of CL by seeing its connectives as conflations.

The philosophical story and the logical facts fit together.

What about if MALL is right?

Then we have an explanation for the success of everything that is NOD-valid;
this is short of full classicality, eg $p, q \nvdash_{\text{NOD}} q$

But it does achieve more than $\gamma^{\pm}$ does:

ϕ	$\vdash_{MALL} \psi$ and $\beta(\psi) = \phi$
$p \vee \neg p$	$p \parr \neg p$
$\neg(p \wedge \neg p)$	$\neg(p \otimes \neg p)$
$(p \to (p \to q)) \to (p \to q)$	$(p \sqsupset (p \sqsupset q)) \multimap (p \sqsupset q)$
$(p \wedge (p \to q)) \to q$	$(p \otimes (p \multimap q)) \multimap q$

Recapturing minimal validities

Question:

Does this work for the theorem fragment?

That is, although $\Gamma \vdash_{\mathrm{CL}} \Delta$ doesn't imply $\Gamma \vdash_{\mathrm{NOD}} \Delta$ in general, does $\vdash_{\mathrm{CL}} A$ imply $\vdash_{\mathrm{NOD}} A$?

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ANSWER: Yes!

That's a special case of a more general fact:

if $[\Gamma \succ \Delta]$ is minimally classically valid,
which is to say that it's classically valid
and that no proper subsequent of it is,
then $[\Gamma \succ \Delta]$ has a proof in NOD.

Since the empty sequent is not classically valid,
the claim about theorems is a special case.

Lemma:

In FULL, we can permute dilutions down:

if there is a proof of $[\Gamma \succ \Delta]$,
then there is one with all dilutions at the end.

It suffices to show that whenever we have a dilution above a non-dilution,
we can replace those two rules with a stretch of proof
that is no longer and has all dilutions at the end.

When no principal formula is diluted in,
we can just swap the order of the rules, eg:

$$\neg L: \dfrac{D: \dfrac{[\Gamma \succ \Delta, A]}{[\Gamma, \Gamma' \succ \Delta, \Delta', A]}}{[\neg A, \Gamma, \Gamma' \succ \Delta, \Delta']} \quad \Rightarrow \quad D: \dfrac{\neg L: \dfrac{[\Gamma \succ \Delta, A]}{[\neg A, \Gamma \succ \Delta]}}{[\neg A, \Gamma, \Gamma' \succ \Delta, \Delta']}$$

When a principal formula is diluted in, and the rule is not $\wedge L^{\otimes}$, we can do all the work just with dilution, eg:

$$\wedge R^{\sqcap}: \quad \frac{[\Gamma, \Gamma' \succ \Delta, \Delta', A] \qquad \text{D:}\ \dfrac{[\Gamma \succ \Delta]}{[\Gamma, \Gamma' \succ \Delta, \Delta', B]}}{[\Gamma, \Gamma' \succ \Delta, \Delta', A \wedge B]}$$

$$\Downarrow$$

$$\text{D:} \quad \frac{[\Gamma \succ \Delta]}{[\Gamma, \Gamma' \succ \Delta, \Delta', A \wedge B]}$$

The same goes for $\wedge L^{\otimes}$, when both conjuncts are diluted in:

$$\begin{array}{rc} & [\Gamma \succ \Delta] \\ \text{D:} & \overline{} \\ & [\Gamma, \Gamma', A, B \succ \Delta, \Delta'] \\ \wedge L^{\otimes}\text{:} & \overline{} \\ & [\Gamma, \Gamma', A \wedge B \succ \Delta, \Delta'] \end{array} \quad \Rightarrow \quad \begin{array}{rc} & [\Gamma \succ \Delta] \\ \text{D:} & \overline{} \\ & [\Gamma, \Gamma', A \wedge B \succ \Delta, \Delta'] \end{array}$$

The fun case is where it's $\wedge L^{\otimes}$ and one conjunct is diluted in:

$$
\begin{array}{rc}
\text{D:} & \dfrac{[\Gamma, A/B \succ \Delta]}{[\Gamma, \Gamma', A, B \succ \Delta, \Delta']} \\
\wedge L^{\otimes}\text{:} & \overline{[\Gamma, \Gamma', A \wedge B \succ \Delta, \Delta']}
\end{array}
\quad \Rightarrow \quad
\begin{array}{rc}
\wedge L^{\sqcap}\text{:} & \dfrac{[\Gamma, A/B \succ \Delta]}{[\Gamma, A \wedge B \succ \Delta]} \\
\text{D:} & \overline{[\Gamma, \Gamma', A \wedge B \succ \Delta, \Delta']}
\end{array}
$$

If $[\Gamma \succ \Delta]$ is minimally classically valid, then $\Gamma \vdash_{\text{NOD}} \Delta$

Proof:

By completeness of FULL, take a FULL proof of $[\Gamma \succ \Delta]$.

By the lemma, all dilutions can be moved to the end.

By soundness and minimality, there must be no dilutions at the end.

So this is a NOD proof.

The converse does not hold; eg there is a NOD proof of $[p, p \succ p \wedge p]$

Summary

- Ambiguity is alleged to give classical recapture for the linear logician
- It doesn't:
  - Ono's translation of atoms is unmotivated
  - Grišin's translation misses key classical theorems
  - Neither translation fits with how disambiguation really works
- Even starting from affine logic, the last problem remains
- Conflation gives a better picture:
  - For the affine logician, full classical logic
  - For the linear logician, partial—but including all minimal validities
  - The logical treatment is built to fit how actual conflation works