Naive validity

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A base system

A formal language

$\mathcal{L}$ is a usual first-order language with equality,
and with three countably infinite stocks $\mathtt{T}_1$, $\mathtt{T}_2$, $\mathtt{T}_3$ of constant terms.

Distinguish two unary predicates: T and V.

Consider only finite arguments $\Gamma \succ \Delta$.
Let the set of all such arguments be $\mathcal{A}$.

$\mathtt{T}_1$ is nothing special; its terms are treated as usual.

Fix a bijection $\tau : \mathtt{T}_2 \to \mathcal{L}$, and a bijection $\nu : \mathtt{T}_3 \to \mathcal{A}$.

Terms from $\mathtt{T}_2$ are distinguished terms for formulas,
and terms from $\mathtt{T}_3$ are distinguished terms for arguments.

For any sentence A,
$\langle A \rangle$ is the term t from $\mathtt{T}_2$ with $\tau(t) = A$.

For any argument $\Gamma \succ \Delta$,
$\langle \Gamma \succ \Delta \rangle$ is the term u from $\mathtt{T}_3$ with $\nu(u) = \Gamma \succ \Delta$.

Depending on τ and ν, such a language can contain paradoxical sentences galore:

Liar

A sentence λ that is $\neg T\langle\lambda\rangle$,

Curry

a sentence κ_A that is $T\langle\kappa\rangle \supset A$,

V-curry

a sentence v_A that is $V\langle v_A \succ A\rangle$,

Pseudo-scotus

a sentence p that is $\neg V\langle \top \succ p \rangle$, a sentence X that is $\neg T\langle \neg V\langle X \succ T\langle \bot \rangle\rangle\rangle$, and so on.

A base system

Validity is unary!

Often validity is treated as a binary predicate on sentences.

For multiple-premise arguments, we are meant to first conjoin the premises into one; similarly disjoin multiple conclusions.

This does violence, though, to the usual notion of validity.
It is arguments that are valid or not.

Worse, it renders certain substantive claims trivial,
like the claim that $A, B \succ C$ is valid iff $A \wedge B \succ C$ is.

The unary validity predicate,
together with names for arguments,
is much more what we should want.

A base system

CLT

For this language, here's a base proof system.

This is meant to register bounds on collections of assertions and denials.

If $\Gamma \succ \Delta$ is derivable, then it's out of bounds to assert everything in Γ and deny everything in Δ.

Structural rules

Id: $$\overline{A \succ A}$$

D: $$\frac{\Gamma \succ \Delta}{\Gamma', \Gamma \succ \Delta, \Delta'}$$

Connective rules

$$\top\text{-drop:}\quad \frac{\top, \Gamma \succ \Delta}{\Gamma \succ \Delta}$$

$$\neg\text{L:}\quad \frac{\Gamma \succ \Delta, A}{\neg A, \Gamma \succ \Delta} \qquad\qquad \neg\text{R:}\quad \frac{A, \Gamma \succ \Delta}{\Gamma \succ \Delta, \neg A}$$

$$\wedge\text{L:}\quad \frac{\Gamma, A, B \succ \Delta}{\Gamma, A \wedge B \succ \Delta} \qquad\qquad \wedge\text{R:}\quad \frac{\Gamma \succ \Delta, A \qquad \Gamma \succ \Delta, B}{\Gamma \succ \Delta, A \wedge B}$$

Quantifier rules

$$\forall\text{L:}\quad \frac{A(t), \Gamma \succ \Delta}{\forall x A(x), \Gamma \succ \Delta} \qquad\qquad \forall\text{R:}\quad \frac{\Gamma \succ \Delta, A(a)}{\Gamma \succ \Delta, \forall x A(x)}$$

t any term; a an eigenvariable $\downarrow$ and any term $\uparrow$.

Equality rules

=-sub: $$\frac{t = u, \Gamma(t)(u) \succ \Delta(t)(u)}{t = u, \Gamma(u)(t) \succ \Delta(u)(t)}$$

=-ref-drop: $$\frac{t = t, \Gamma \succ \Delta}{\Gamma \succ \Delta}$$

$\Gamma(t)(u)$ has some (maybe 0) occurrences of t and u selected.

So far: exactly classical first-order logic with equality (CFOLE).

All these rules apply to all vocabulary—so CFOLE does as well.

Truth rules

$$TL: \quad \frac{A, \Gamma \succ \Delta}{T\langle A \rangle, \Gamma \succ \Delta} \qquad\qquad TR: \quad \frac{\Gamma \succ \Delta, A}{\Gamma \succ \Delta, T\langle A \rangle}$$

A a sentence.

These truth rules conservatively extend(*) the base system.

Paradoxes are dissolved because cut is no longer admissible.

$$\text{Cut:} \quad \frac{\Gamma \succ \Delta, A \qquad A, \Gamma \succ \Delta}{\Gamma \succ \Delta}$$

In terms of bounds, cut registers a certain optimism:
if $\Gamma \succ \Delta$ is in bounds, then either
$\Gamma \succ \Delta, A$ is or $A, \Gamma \succ \Delta$ is.

Whatever A is, there's some in-bounds way to go on with it.

I'm not so optimistic.

Validity rules

What is validity?

Often, (relatives of) these rules are given for validity:

VD: $$\frac{}{\Gamma, V\langle\Gamma \succ \Delta\rangle \succ \Delta}$$

VP: $$\frac{\Gamma \succ \Delta}{\succ V\langle\Gamma \succ \Delta\rangle}$$

Often, (relatives of) these rules are given for validity:

VD: $$\dfrac{}{\Gamma, V\langle \Gamma \succ \Delta \rangle \succ \Delta}$$

VR: $$\dfrac{\Gamma_V, \Gamma \succ \Delta}{\Gamma_V \succ V\langle \Gamma \succ \Delta \rangle}$$

Often, (relatives of) these rules are given for validity:

VD: $$\frac{}{\Gamma, V\langle\Gamma \succ \Delta\rangle \succ \Delta}$$

VR: $$\frac{\Gamma_V, \Gamma \succ \Delta, \Delta_V}{\Gamma_V \succ V\langle\Gamma \succ \Delta\rangle, \Delta_V}$$

Are these rules any good?

It depends on what validity is.

But here I’ve taken a stand already:
validity is a matter of bounds on collections
of assertions and denials.

Call these collections positions.

Validity rules

VD

VD—initial sequents $\Gamma, V\langle\Gamma \succ \Delta\rangle \succ \Delta$—says
that it is out of bounds to assert that a position is out of bounds
when you have taken up that very position.

When $\Gamma \succ \Delta$ really is out of bounds, there's not much to see here.

The fun case is when it isn't.

The question around VD is:
how much confidence do the bounds have in themselves?

Suppose I think the bounds have gone wrong:
they forbid me to assert p, but p is right.

Then I might assert both p and $V\langle p \succ \rangle$.
VD rules this out. Should it?

Validity rules

VP

VP says that if $\Gamma \succ \Delta$ is actually out of bounds,
then it's out of bounds to deny this.

Why should this be?
There are presumably lots of mistaken denials that are in bounds.

The question around VP is:
how transparent are the bounds to themselves?

Validity rules

Internalization

I’m suspicious of both VD and VP/VR.

I don’t see why the bounds should be either
confident in themselves in the way VD registers,
or transparent to themselves in the way VP/VR registers.

But it’s still worth seeing that even if they are,
there is no problem.

Indeed, in the presence of VD and VR,
the calculus has the interesting property of internalization,
identified by Barrio, Rosenblatt, & Tajer:

For every rule $\Gamma_1 \succ \Delta_1, \ldots, \Gamma_n \succ \Delta_n \Rightarrow \Gamma \succ \Delta$ derivable in the calculus, the sequent $V\langle\Gamma_1 \succ \Delta_1\rangle, \ldots, V\langle\Gamma_n \succ \Delta_n\rangle \succ V\langle\Gamma \succ \Delta\rangle$ is also derivable.

(and so is $\succ V\langle\Gamma_1 \succ \Delta_1\rangle \wedge \ldots \wedge V\langle\Gamma_n \succ \Delta_n\rangle \supset V\langle\Gamma \succ \Delta\rangle$)

Nontriviality

Conservative extension($\star$)

But of course a trivial calculus
—one that derives every sequent—
also has the internalization property.

How can we be sure that the paradoxes
have not blown this calculus up?

The first proof is simple: count a predicate as occurring* in a sequent iff either it occurs in the usual sense, or else it occurs in the usual sense in A, and $\langle A \rangle$ occurs* in the sequent.

No rule can take us from a sequent in which V occurs* to one in which it does not.

So once Vs get into a proof, they stay there.
Both VD and VP always put a V in.

So if a sequent $\Gamma \succ \Delta$ in which V does not occur* is derivable,
it's derivable without the use of VD or VP.

Thus, $p \succ q$ (and many other sequents!) not derivable.
The paradoxes have not blown up.

Nontriviality

Beyond conservative extension($\star$)

That's a good start.

But it's compatible with conservative extension(⋆)
that every sequent in which V does occur⋆ is provable.

That wouldn't be near as bad as being trivial,
but it would still be pretty bad.

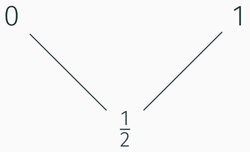
So more work is needed.

Suitable models are strong Kleene models
with domains that contain the language and all arguments,
and which interpret all $\langle A \rangle$ as A and $\langle \Gamma \succ \Delta \rangle$ as $\Gamma \succ \Delta$.

A model M is a countermodel to $\Gamma \succ \Delta$, written $M \lightning \Gamma \succ \Delta$,
iff $M(\Gamma) = 1$ and $M(\Delta) = 0$.

CL alone is sound for these.

Take the information ordering $\sqsubseteq$ on truth-in-a-model-values:



Extend this pointwise to order interpretations of predicates,
and then to order models (which must agree on domain and terms).

Note: if $M \lightning \Gamma \succ \Delta$ and $M \sqsubseteq M'$, then $M' \lightning \Gamma \succ \Delta$.

$M \approx_{TV} M'$ iff M and M' match on everything except maybe the interpretations of T applied to sentences and V applied to arguments.

$M \sqsubseteq_{TV} M'$ iff $M \approx_{TV} M'$ and $M \sqsubseteq M'$.

Given any suitable model M, define suitable $j(M)$:

- $j(M)(T)(A) = M(A)$
- $j(M)(V)(\Gamma \succ \Delta) = 0$ iff $M \lightning \Gamma \succ \Delta$
- $j(M)(V)(\Gamma \succ \Delta) = 1$ iff there is no $M' \sqsupseteq_{TV} M$ st $M' \lightning \Gamma \succ \Delta$
- $j(M)(V)(\Gamma \succ \Delta) = \frac{1}{2}$ otherwise
- $j(M)$ matches M everywhere else

Always, $M \approx_{TV} j(M)$.

Now fix a suitable M_0 with $M_0 \sqsubseteq_{TV} j(M_0)$, and define M^i for ordinals i:

$$M^0 = M_0$$
$$M^{i+1} = j(M^i)$$
$$M^{\lim} = \max_{k<\lim} M^k$$

Fact: this is a chain, with $M^i \sqsubseteq_{TV} M^j$ if $i \leq j$.
Fact: this reaches a fixed point, a model M with $M = j(M)$.

Rules for semantic vocab:

These are all sound for fixed points:

$$TL: \quad \frac{A, \Gamma \succ \Delta}{T\langle A \rangle, \Gamma \succ \Delta} \qquad\qquad TR: \quad \frac{\Gamma \succ \Delta, A}{\Gamma \succ \Delta, T\langle A \rangle}$$

$$VD: \quad \frac{}{\Gamma, V\langle \Gamma \succ \Delta \rangle \succ \Delta} \qquad\qquad VR: \quad \frac{\Gamma_V, \Gamma \succ \Delta, \Delta_V}{\Gamma_V \succ V\langle \Gamma \succ \Delta \rangle, \Delta_V}$$

Nontriviality

What’s not provable

So if there's a countermodel M to $\Gamma \succ \Delta$ with $M \sqsubseteq_{TV} j(M)$, then there is a fixed point $\sqsupseteq_{TV} M$.

It, too, must be a countermodel to $\Gamma \succ \Delta$.

So $\Gamma \succ \Delta$ is not provable in the full system.

Thus eg $p, V\langle r \succ q\rangle \succ r, V\langle q \succ r\rangle$ is not provable:

Take the premises to 1 and the conclusions to 0, q to 1, and all other semantic cases to $\frac{1}{2}$.

Such a model is a countermodel below its own jump.

For any sequent $\Gamma \succ \Delta$ in which V does not occur*, it is provable in the full system iff in CLT.

When V does occur*, it's more case by case.

Key question: is there a countermodel M to $\Gamma \succ \Delta$ with $M \sqsubseteq_{TV} j(M)$?